# Exercise 7

## I. Action of the affine Weyl group

1. Calculate explicitly $s_{\hat{\alpha}}\hat{\lambda}$ with $\hat{\lambda} = (\lambda; k; n)$ and $\hat{\alpha} = (\alpha; 0; m)$.

2. Verify the above result by calculating $s_{\hat{\alpha}}\hat{\alpha}$.

3. Verify $(t_{\alpha^\vee})(t_{\beta^\vee}) = t_{\alpha^\vee+\beta^\vee}$.

4. Calculate $(t_{\alpha^\vee})^m$.

5. Verify $w(t_{\alpha^\vee})w^{-1} = t_{w\alpha^\vee}$.

6. Calculate $(s_{\hat{\alpha}}\hat{\lambda}, s_{\hat{\alpha}}\hat{\lambda})$.

## II. The affine Weyl group of $\hat{A}_1$

1. Give the action of the Weyl reflections with respect to the simple roots $s_i$ on a weight $\hat{\lambda} = [\lambda_0, \lambda_1]$.

2. Assume that the level of $\hat{\lambda}$ is $k$. Express the action of the Weyl reflections on $\hat{\lambda}$ in terms of $k$.

3. Show that the basic translation operator of $\widehat{W}_{\hat{A}_1}$ is $t_{\alpha_1^\vee} = s_0 s_1$.

4. Draw the weight spaces level by level and draw the Weyl chambers.

## III. The affine Weyl group of $\hat{A}_2$

1. Give the action of the Weyl reflections with respect to the simple roots $s_i$ on a weight $\hat{\lambda} = [\lambda_0, \lambda_1, \lambda_2]$.

2. Assume again that the level of $\hat{\lambda}$ is $k$ and express $\lambda_0$ in terms of $k$. What is the function of the elements $s_2 s_0 s_2 s_1$ and $s_1 s_0 s_1 s_2$?

3. Draw the affine Weyl chambers for the level $k = 2$.

## IV. Symmetries of the affine Dynkin diagrams

1. List all the symmetry transformations that leave the Dynkin diagrams of the untwisted affine Lie algebras invariant (draw on the diagrams).

2. Compare the symmetry transformations of the Dynkin diagrams of $A_r$, $r > 1$ and $\hat{A}_r$, $r > 1$. What is $\mathcal{O}(\hat{A}_r)$?

3. Compare the symmetry transformations of the Dynkin diagrams of $E_6$ and $\hat{E}_6$. What is $\mathcal{O}(\hat{E}_6)$?

**V. Outer automorphsims of $\hat{A}_1$**

1. What is the outer automorphism group of $\hat{A}_1$?
2. How does it act on a general weight?
3. What is the element $w_a$ for the basic element $a$?

**VI. Outer automorphsims of $\hat{A}_2$**

1. How does the basic element $a$ of $\mathcal{O}(\hat{A}_2)$ act on the fundamental weights?
2. How does it act on the Dynkin labels of a general weight $\hat{\lambda}$?
3. Construct the element $w_a$.

**VII. Outer automorphsims of $\hat{A}_r$**

Based on the last two exercises, what is the general result for $w_a$ for any $r$?