CHAPTER 12

$\mathcal{N} = (2,2)$ Supersymmetry

In our discussion of supersymmetric QFTs in dimensions 0 and 1, we have presented actions which possess fermionic symmetries. We did not present any systematic discussion of how one arrives at such actions. We will remedy this gap in this section and the next. We develop the notion of superspace which, in addition to the usual bosonic coordinates, contains fermionic coordinates (as many as the number of supersymmetries). We will also generalize the notion of fields to superfields. Supersymmetry is realized on the superspace by translations in the fermionic directions. Writing down actions which are coordinate invariant in the superspace sense will thus naturally lead to supersymmetric actions.

Here we will mainly consider supersymmetric field theories in 1+1 dimensions with four real supercharges (or two complex supercharges), two with positive chirality and two with negative chirality. This is called $\mathcal{N} = (2,2)$ supersymmetry, and is relevant for mirror symmetry. By reduction to 1 and 0 dimensions, one obtains the actions discussed in the previous sections for the case with four supercharges. One can also develop superspace techniques for the case with two supercharges. That will be recorded in Appendix 12.5.

12.1. Superfield Formalism

We start our discussion by providing a systematic way to obtain supersymmetric Lagrangians. This involves introducing superspace and superfields.

12.1.1. Superspace and Superfields. We consider a field theory on $\mathbb{R}^2$ with time and space coordinates

$$x^0 = t, \ x^1 = s. \tag{12.1}$$

We take the flat Minkowski metric $\eta_{00} = -1$, $\eta_{11} = 1$ and $\eta_{01} = 0$. Besides these bosonic coordinates we introduce four fermionic coordinates

$$\theta^+,\, \theta^-,\, \overline{\theta}^+,\, \overline{\theta}^-. \tag{12.2}$$

These are complex fermionic coordinates which are related to each other by complex conjugation, $(\theta^\pm)^* = \overline{\theta}^\pm$. The indices $\pm$ stand for the spin (or chirality) under a Lorentz transformation. Namely, a Lorentz transformation acts on the bosonic and fermionic coordinates as

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \to \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \tag{12.3}$$

$$\theta^\pm \;\to\; \mathrm{e}^{\pm\gamma/2}\theta^\pm,\; \overline{\theta}^\pm \;\to\; \mathrm{e}^{\pm\gamma/2}\overline{\theta}^\pm. \tag{12.4}$$

The fermionic coordinates anti-commute with each other, $\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$, $\overline{\theta}^\alpha\overline{\theta}^\beta = -\overline{\theta}^\beta\overline{\theta}^\alpha$, and $\theta^\alpha\overline{\theta}^\beta = -\overline{\theta}^\beta\theta^\alpha$. The $(2,2)$ *superspace* is the space with the coordinates x^0, x^1, $\theta^\pm$, $\overline{\theta}^\pm$.

Superfields are functions defined on the superspace. They can be Taylor expanded in monomials in $\theta^\pm$ and $\overline{\theta}^\pm$.

$$\begin{aligned} \mathcal{F}(x^0,x^1,\theta^+,\theta^-,\overline{\theta}^+,\overline{\theta}^-) =& f_0(x^0,x^1) + \theta^+ f_+(x^0,x^1) \\ &+\theta^- f_-(x^0,x^1) + \overline{\theta}^+ f'_+(x^0,x^1) \\ &+\overline{\theta}^- f'_-(x^0,x^1) + \theta^+\theta^- f_{+-}(x^0,x^1) + \cdots . \end{aligned} \tag{12.5}$$

(Superfields are to supersymmetry what N-vector fields are to $SO(N)$ symmetry – a convenient organizational scheme.) Since any of the fermionic coordinates squares to zero, $(\theta^\pm)^2 = (\overline{\theta}^\pm)^2 = 0$, there are at most $2^4 = 16$ nonzero terms in the expansion. A superfield Φ is bosonic if $[\theta^\alpha, \Phi] = 0$ and is fermionic if $\{\theta^\alpha, \Phi\} = 0$. We introduce some differential operators on the superspace,

$$\mathcal{Q}_\pm \;=\; \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\,\partial_\pm, \tag{12.6}$$

$$\overline{\mathcal{Q}}_\pm \;=\; -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\,\partial_\pm. \tag{12.7}$$

Here $\partial_\pm$ are differentiations by $x^\pm := x^0 \pm x^1$:

$$\partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{12.8}$$

These differential operators satisfy the anti-commutation relations

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -2i\partial_\pm, \tag{12.9}$$

with all other anti-commutators vanishing. We define another set of differential operators

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\,\partial_\pm, \tag{12.10}$$

$$\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\,\partial_\pm, \tag{12.11}$$

which anti-commute with $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$, i.e., $\{D_\pm, \mathcal{Q}_\pm\} = 0$, etc. These obey similar anti-commutation relations

$$\{D_\pm, \overline{D}_\pm\} = 2i\partial_\pm, \tag{12.12}$$

with all other anti-commutators vanishing.

Exercise 12.1.1. *We have discussed the notion of superspace adapted to the signature* $(1,1)$. *Generalize this to the Euclidean signature. In particular show that the* $\pm$ *index on* $x^\pm$ *and* $\theta^\pm, \overline{\theta}^\pm$ *distinguishes holomorphic versus anti-holomorphic supercoordinates.*

Vector R-rotations and *axial R-rotations* of a superfield are defined by

$$\mathrm{e}^{i\alpha F_V} : \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto \mathrm{e}^{i\alpha q_V}\mathcal{F}(x^\mu,\, \mathrm{e}^{-i\alpha}\theta^\pm,\, \mathrm{e}^{i\alpha}\overline{\theta}^\pm), \tag{12.13}$$

$$\mathrm{e}^{i\beta F_A} : \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto \mathrm{e}^{i\beta q_A}\mathcal{F}(x^\mu,\, \mathrm{e}^{\mp i\beta}\theta^\pm,\, \mathrm{e}^{\pm i\beta}\overline{\theta}^\pm), \tag{12.14}$$

where q_V and q_A are numbers called vector R-charge and axial R-charge of $\mathcal{F}$. The transformations given by Eqs. (12.13)–(12.14) induce transformations of the constituent fields of $\mathcal{F}$.

A *chiral superfield* Φ is a superfield that satisfies the equations

$$\overline{D}_\pm\Phi = 0. \tag{12.15}$$

If Φ_1 and Φ_2 are chiral superfields, the product $\Phi_1\Phi_2$ is also a chiral superfield. A general chiral superfield Φ has the form

$$\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm), \tag{12.16}$$

where $y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm$. The complex conjugate of a chiral superfield Φ obeys the condition

$$D_\pm\overline{\Phi} = 0 \tag{12.17}$$

and is called an *anti-chiral superfield.*

Exercise 12.1.2. *Show that a chiral superfield can be expanded as shown in Eq. (12.16).*

A *twisted chiral superfield* U is a superfield that satisfies

$$\overline{D}_+ U = D_- U = 0. \tag{12.18}$$

If U_1 and U_2 are twisted chiral superfields, the product $U_1 U_2$ is also a twisted chiral superfield. A general twisted chiral superfield U has the form

$$U(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \upsilon(\widetilde{y}^\pm) + \theta^+ \overline{\chi}_+(\widetilde{y}^\pm) + \overline{\theta}^- \chi_-(\widetilde{y}^\pm) + \theta^+ \overline{\theta}^- E(\widetilde{y}^\pm), \tag{12.19}$$

where $\widetilde{y}^\pm = x^\pm \mp i\theta^\pm \overline{\theta}^\pm$. The complex conjugate $\overline{U}$ of a twisted chiral superfield U obeys the condition

$$D_+ \overline{U} = \overline{D}_- \overline{U} = 0 \tag{12.20}$$

and is called a *twisted anti-chiral superfield.*

12.1.2. Supersymmetric Actions. We now construct action functionals of superfields that are invariant under the transformation

$$\delta = \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \overline{\epsilon}_+ \overline{\mathcal{Q}}_- + \overline{\epsilon}_- \overline{\mathcal{Q}}_+. \tag{12.21}$$

Let us first consider the functional of the superfields $\mathcal{F}_i$ of the form

$$\int d^2x\, d^4\theta\, K(\mathcal{F}_i) = \int d^2x\, d\theta^+ d\theta^- d\overline{\theta}^- d\overline{\theta}^+\, K(\mathcal{F}_i), \tag{12.22}$$

where $K(-)$ is an arbitrary differentiable function of the $\mathcal{F}_i$'s. This is invariant under the variation δ. For example, let us look at the term proportional to ϵ_+;

$$\int d^2x d^4\theta\, \epsilon_+ (\mathcal{Q}_- \mathcal{F}_i) \frac{\partial K}{\partial \mathcal{F}_i} = \int d^2x d^4\theta\, \epsilon_- \left(\frac{\partial}{\partial \theta^-} + i\overline{\theta}^- \partial_- \right) K(\mathcal{F}_i). \tag{12.23}$$

The integration over $d^4\theta$ is nonzero only if we have $\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$. Therefore the first term is zero since the integrand does not have θ^- because of the derivative $\partial/\partial\theta^-$. The second term is a total derivative and vanishes after integration over d^2x. Vanishing of the coefficients of ϵ_+ and $\overline{\epsilon}_\pm$ can be seen in a similar way. The functional of the form shown in Eq. (12.22) is called a *D-term.*

We next consider the functional of chiral superfields Φ_i of the form

$$\int d^2x d^2\theta\, W(\Phi_i) = \int d^2x\, d\theta^- d\theta^+\, W(\Phi_i)\Big|_{\overline{\theta}^\pm = 0}, \tag{12.24}$$

where $W(\Phi_i)$ is a holomorphic function of the Φ_i's. This is also invariant under the variation δ. Let us first look at the coefficient of $\epsilon_\pm$:

$$\pm \int d^2x\, d\theta^- d\theta^+\, \epsilon_\pm \left(\frac{\partial}{\partial \theta^\mp} + i\overline{\theta}^\mp \partial_\mp \right) W(\Phi_i) \Big|_{\overline{\theta}^\pm = 0} . \tag{12.25}$$

The first term vanishes for the standard reason. The second term vanishes because we put $\overline{\theta}^\pm = 0$ (or since it is a total derivative). Let us next look at the coefficient of $\overline{\epsilon}_\pm$. For this we note that $\overline{\mathcal{Q}}_\pm = \overline{D}_\pm - 2i\theta^\pm \partial_\pm$. Then the variation is

$$\mp \int d^2x\, d\theta^- d\theta^+\, \overline{\epsilon}_\pm \left(\overline{D}_\mp - 2i\theta^\pm \partial_\pm \right) W(\Phi_i) \Big|_{\overline{\theta}^\pm = 0} . \tag{12.26}$$

The first term in the integrand $\overline{D}_\mp W(\Phi_i)$ is zero because Φ_i are chiral superfields and $W(\Phi_i)$ is a holomorphic function (it does not contain $\overline{\Phi}_i$). The second integral vanishes because it is a total derivative in x^μ. The functional of the form shown in Eq. (12.24) is called an *F-term*.

We finally consider the functional of twisted chiral superfields U_i of the form

$$\int d^2x\, d^2\widetilde{\theta}\, \widetilde{W}(U_i) = \int d^2x\, d\overline{\theta}^-\, d\theta^+\, \widetilde{W}(U_i) \Big|_{\overline{\theta}^+ = \theta^- = 0}, \tag{12.27}$$

where $\widetilde{W}(U_i)$ is a holomorphic function of the U_i's. By a similar argument as in the case of the F-term, one can see that this functional is invariant under δ. The functional of the form shown in Eq. (12.27) is called a *twisted F-term*.

12.1.3. Some Superfield Calculus. We present some calculus on superspace, some of which will be used in later sections. However, the reader can skip these exercises in the first reading and return when they are needed.

The basic element of the superfield calculus is the analogue of Poincaré's lemma in the ordinary calculus. Suppose $\mathcal{F}$ is a superfield that decays rapidly at infinity in (x^0, x^1)-space. Then

Lemma 12.1.1 (Poincaré's Lemma). *$D_+\mathcal{F} = 0$ implies $\mathcal{F} = D_+\mathcal{G}$ for some superfield $\mathcal{G}$. The same is true for the differential operators D_-, $\overline{D}_+$ and $\overline{D}_-$.*

Proof. $D_+\mathcal{F} = 0$ implies $\overline{D}_+ D_+ \mathcal{F} = 0$. Using the anti-commutation relation from Eq. (12.12), we find $2i\partial_+\mathcal{F} = D_+\overline{D}_+\mathcal{F}$. Since $\mathcal{F}$ decays

rapidly at infinity, we can integrate this relation as

$$2i\mathcal{F} = \int_{-\infty}^{x^+} D_+\overline{D}_+\mathcal{F}\, dx'^+ = D_+ \int_{-\infty}^{x^+} \overline{D}_+\mathcal{F}\, dx'^+. \tag{12.28}$$

We thus obtain $\mathcal{F} = D_+\mathcal{G}$ where $\mathcal{G}$ is the superfield $\frac{1}{2i}\int_{-\infty}^{x^+} \overline{D}_+\mathcal{F}dx'^+$. This is what we wanted to show. □

The reasoning used in the proof is sufficient to show the following.
1. $\overline{D}_+\overline{D}_-\mathcal{F} = 0$ implies $\mathcal{F} = \mathcal{G}_+ + \mathcal{G}_-$ for some superfields $\mathcal{G}_\pm$ such that $\overline{D}_+\mathcal{G}_+ = 0$ and $\overline{D}_-\mathcal{G}_- = 0$. Similar results hold for D_+D_-, $\overline{D}_+D_-$ and $D_+\overline{D}_-$.
2. A chiral superfield Φ can be written as $\Phi = \overline{D}_+\overline{D}_-\mathcal{E}$ for some superfield $\mathcal{E}$. If U is a twisted chiral superfield it can be written as $U = \overline{D}_+D_-\mathcal{V}$.
3. $\overline{D}_+\overline{D}_-\mathcal{F} = D_+D_-\mathcal{F} = 0$ implies $\mathcal{F} = U_1 + \overline{U_2}$ for some twisted chiral superfields U_i. For the equation $\overline{D}_+D_-\mathcal{F} = D_+\overline{D}_-\mathcal{F} = 0$, we have $\mathcal{F} = \Phi_1 + \overline{\Phi_2}$ for some chiral superfields Φ_i.

Exercise 12.1.3. *Prove the above statements.*

Let us consider the integral

$$\int d^2x\, d^4\theta AB \tag{12.29}$$

where A and B are arbitrary superfields. It is easy to see that the extremum of this integral with respect to the variation of A is attained only by $B = 0$.

Exercise 12.1.4. *Show that if A is restricted to be a chiral superfield, then the extrema are attained by B with $\overline{D}_+\overline{D}_-B = 0$.*

12.2. Basic Examples

Here we present basic examples of classical $(2,2)$ supersymmetric field theories. One is a theory of a single chiral superfield and the other is a theory of a single twisted chiral superfield.

12.2.1. Theory of a Chiral Superfield. We first consider a supersymmetric action for a single chiral superfield Φ. As noted above, the superfield Φ has the following θ-expansion

$$\begin{aligned}\Phi &= \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^-F(y^\pm)\\ &= \phi - i\theta^+\overline{\theta}^+\partial_+\phi - i\theta^-\overline{\theta}^-\partial_-\phi - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\phi\\ &\quad +\theta^+\psi_+ - i\theta^+\theta^-\overline{\theta}^-\partial_-\psi_+ + \theta^-\psi_- - i\theta^-\theta^+\overline{\theta}^+\partial_+\psi_- + \theta^+\theta^-F,\end{aligned} \tag{12.30}$$

where in the last equality we have further expanded $y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm$ at $x^\pm$. The θ-expansion of the anti-chiral superfield $\overline{\Phi}$ is easily obtained by complex conjugation of Eq. (12.30);

$$\begin{aligned}\overline{\Phi} &= \overline{\phi} + i\theta^+\overline{\theta}^+\partial_+\overline{\phi} + i\theta^-\overline{\theta}^-\partial_-\overline{\phi} - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\overline{\phi} \\ &\quad -\overline{\theta}^+\overline{\psi}_+ - i\overline{\theta}^+\theta^-\overline{\theta}^-\partial_-\overline{\psi}_+ - \overline{\theta}^-\overline{\psi}_- - i\overline{\theta}^-\theta^+\overline{\theta}^+\partial_+\overline{\psi}_- + \overline{\theta}^-\overline{\theta}^+\overline{F}.\end{aligned} \tag{12.31}$$

Note that $(\psi_1\psi_2)^* = \psi_2^*\psi_1^*$ for fermionic variables/coordinates.

Now let us compute the D-term

$$S_{\text{kin}} = \int d^2x\, d^4\theta\, \overline{\Phi}\Phi. \tag{12.32}$$

The integration $\int d^4\theta\, \overline{\Phi}\Phi$ amounts to extracting the coefficient of $\theta^4 = \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+$ in the θ-expansion of $\overline{\Phi}\Phi$. By a straightforward computation we have

$$\begin{aligned}\overline{\Phi}\Phi\Big|_{\theta^4} &= -\overline{\phi}\partial_+\partial_-\phi + \partial_+\overline{\phi}\partial_-\phi + \partial_-\overline{\phi}\partial_+\phi - \partial_+\partial_-\overline{\phi}\phi \\ &\quad + i\overline{\psi}_+\partial_-\psi_+ - i\partial_-\overline{\psi}_+\psi_+ + i\overline{\psi}_-\partial_+\psi_- - i\partial_+\overline{\psi}_-\psi_- + |F|^2.\end{aligned} \tag{12.33}$$

Here again, the derivatives of fields appear due to the changing of variables from y to x and doing the Taylor expansion around $\theta = 0$. By partial integration, the action takes the form

(12.34)

$$S_{\text{kin}} = \int d^2x \left(|\partial_0\phi|^2 - |\partial_1\phi|^2 + i\overline{\psi}_-(\partial_0 + \partial_1)\psi_- + i\overline{\psi}_+(\partial_0 - \partial_1)\psi_+ + |F|^2\right).$$

Thus, we have obtained the standard kinetic term for the complex scalar field ϕ and the Dirac fermion fields $\psi_\pm$, $\overline{\psi}_\pm$. Note also that the field F has no kinetic term. Such a field is often called an *auxiliary field* (such as in the path-integral derivation of T-duality). Next let us compute the F-term

$$S_W = \int d^2x\, d^2\theta\, W(\Phi) \,+\, c.c. \tag{12.35}$$

for a holomorphic function $W(\Phi)$ of Φ. This holomorphic function is called a *superpotential*. The integral $\int d^2\theta W(\Phi)$ amounts to extracting the coefficient of $\theta^2 = \theta^+\theta^-$ in the θ-expansion of $W(\Phi)$. It is straightforward to see

$$W(\Phi)\Big|_{\theta^2} = W'(\phi)F - W''(\phi)\psi_+\psi_-. \tag{12.36}$$

Thus, the F-term is

(12.37)
$$S_W = \int d^2x \left(W'(\phi)F - W''(\phi)\psi_+\psi_- + \overline{W}'(\overline{\phi})\overline{F} - \overline{W}''(\overline{\phi})\overline{\psi}_-\overline{\psi}_+ \right).$$

Now let us consider the sum of S_{kin} and S_W as the total action;

$$S = S_{\text{kin}} + S_W. \tag{12.38}$$

By completing the square of F, we obtain the following action

$$\begin{aligned} S = \int d^2x \Big(& |\partial_0\phi|^2 - |\partial_1\phi|^2 - |W'(\phi)|^2 + i\overline{\psi}_-(\partial_0 + \partial_1)\psi_- \\ & + i\overline{\psi}_+(\partial_0 - \partial_1)\psi_+ - W''(\phi)\psi_+\psi_- \\ & - \overline{W}''(\overline{\phi})\overline{\psi}_-\overline{\psi}_+ + |F + \overline{W}'(\overline{\phi})|^2 \Big). \end{aligned} \tag{12.39}$$

Note that the last term $|F + \overline{W}'(\overline{\phi})|^2$ can be eliminated by solving the equation of motion as

$$F = -\overline{W}'(\overline{\phi}). \tag{12.40}$$

Setting F to this value can also be viewed as a result of integrating out F in the path-integral. To summarize, we have obtained the action for the scalar ϕ and the Dirac fermion $\psi_\pm$, $\overline{\psi}_\pm$ with a potential $|W'(\phi)|^2$ for ϕ and the fermion mass term (or Yukawa interaction) $W''(\phi)\psi_+\psi_-$.

By construction, the action is invariant under the variation δ from Eq. (12.21). This variation on the superfield Φ can actually be identified as a certain variation of the ordinary fields ϕ, $\psi_\pm$, $\overline{\psi}_\pm$ and F — the component fields of Φ. This is obvious if the superfield $\mathcal{F}$ is unconstrained. Simply *define* each coefficient field of the θ-expansion of $\delta\mathcal{F}$ as the variation of the corresponding coefficient field of the θ-expansion of $\mathcal{F}$. For example, for the general superfield given in Eq. (12.5), the δ-variation yields

$$\delta\mathcal{F} = \epsilon_+ f_- - \epsilon_- f_+ + \overline{\epsilon}_- f' + \overline{\epsilon}_+ f'_- + \theta^+(\cdots) + \cdots, \tag{12.41}$$

Then we *define* $\delta f_0 = \epsilon_+ f_- - \epsilon_- f_+ + \overline{\epsilon}_- f' + \overline{\epsilon}_+ f'_-$, $\delta f_+ = (\cdots)$, etc. A chiral superfield is not an arbitrary superfield but rather satisfies $\overline{D}_\pm\Phi = 0$. The last condition means that there are relations between the coefficient fields, as can be explicitly seen in Eq. (12.30). Thus, it is not obvious whether the variation δ of Φ can be represented by a variation of the component fields of Φ. However, this is actually the case. The key point is that the differential

operators $\mathcal{Q}_\pm$, $\overline{\mathcal{Q}}_\pm$ anti-commute with $\overline{D}_\pm$ (and also with $D_\pm$) and hence the variation $\delta\Phi$ is also a chiral superfield

$$\overline{D}_\pm \delta\Phi = \delta\overline{D}_\pm \Phi = 0. \tag{12.42}$$

Indeed, one can explicitly show that the variation in question is given by

$$\begin{aligned} \delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+, \\ \delta\psi_\pm &= \pm 2i\overline{\epsilon}_\mp \partial_\pm \phi + \epsilon_\pm F, \\ \delta F &= -2i\overline{\epsilon}_+\partial_-\psi_+ - 2i\overline{\epsilon}_-\partial_+\psi_-. \end{aligned} \tag{12.43}$$

One can replace F by its equation of motion and write a supersymmetry variation of the ϕ and ψ fields alone (true after imposing the equations of motion). One can explicitly check (though it is not necessary) that the action S (or S_{kin} and S_W) is invariant under this variation of the component fields. By the anti-commutation relations from Eq. (12.9), the variations for different parameters ϵ_1 and ϵ_2 satisfy the commutation relation

$$[\delta_1, \delta_2] = 2i(\epsilon_{1-}\overline{\epsilon}_{2-} - \epsilon_{2-}\overline{\epsilon}_{1-})\partial_+ + 2i(\epsilon_{1+}\overline{\epsilon}_{2+} - \epsilon_{2+}\overline{\epsilon}_{1+})\partial_-. \tag{12.44}$$

This is a relation in quantum mechanics that generalizes the supersymmetry relation given by Eq. (10.77). We refer to this situation by saying *the classical field theory with the action given by Eq. (12.39) has* $\mathcal{N} = (2,2)$ *supersymmetry generated by Eq. (12.43).*

Since the classical system has a symmetry, one can find via the Noether procedure the conserved currents and conserved charges. The conserved currents are

$$G^0_\pm = 2\partial_\pm\overline{\phi}\,\psi_\pm \mp i\overline{\psi}_\mp \overline{W}'(\overline{\phi}), \tag{12.45}$$

$$G^1_\pm = \mp 2\partial_\pm\overline{\phi}\psi_\pm - i\overline{\psi}_\mp \overline{W}'(\overline{\phi}), \tag{12.46}$$

$$\overline{G}^0_\pm = 2\overline{\psi}_\pm\partial_\pm\phi \pm i\psi_\mp W'(\phi), \tag{12.47}$$

$$\overline{G}^1_\pm = \mp 2\overline{\psi}_\pm\partial_\pm\phi \pm i\psi_\mp W'(\phi), \tag{12.48}$$

and the conserved charges (supercharges) are

$$Q_\pm = \int dx^1\, G^0_\pm,\ \overline{Q}_\pm = \int dx^1\, \overline{G}^0_\pm. \tag{12.49}$$

These charges transform as spinors

$$Q_\pm \mapsto \mathrm{e}^{\mp\gamma/2} Q_\pm,\ \overline{Q}_\pm \mapsto \mathrm{e}^{\mp\gamma/2}\overline{Q}_\pm, \tag{12.50}$$

under Lorentz transformation as shown by Eq. (12.3).

EXERCISE 12.2.1. *Verify the expressions from Eq. (12.49) for the supercharges.*

This system has more global symmetries. First, by assigning axial R-charge 0 for Φ, the action is invariant under axial R-rotation;

$$\Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm) \mapsto \Phi(x^\pm, \mathrm{e}^{\mp i\alpha}\theta^\pm, \mathrm{e}^{\pm i\alpha}\overline{\theta}^\pm). \tag{12.51}$$

This is obvious in the superspace expressions given by Eq. (12.32) and Eq. (12.35): the products θ^4 and θ^2 are both invariant under the axial rotation. Thus, the system has an axial R-symmetry. The axial rotation of the superfields can be realized as a transformation of the component fields (this can also be understood by looking at the commutation relation of $\overline{D}_\pm$ and the axial rotation). The transformation is given by

$$\phi \mapsto \phi, \ \psi_\pm \mapsto \mathrm{e}^{\mp i\alpha}\psi_\pm. \tag{12.52}$$

The corresponding current is given by

$$J^0_A = \overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-, \tag{12.53}$$

$$J^1_A = -\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-, \tag{12.54}$$

and the conserved charge is

$$F_A = \int J^0_A \, dx^1. \tag{12.55}$$

We note that the axial R-rotation rotates the supercharges as

$$Q_\pm \mapsto \mathrm{e}^{\mp i\alpha}Q_\pm, \ \overline{Q}_\pm \mapsto \mathrm{e}^{\pm i\alpha}\overline{Q}_\pm. \tag{12.56}$$

Second, depending on the form of the superpotential $W(\Phi)$, the system is also invariant under the vector R-rotation. Since θ^4 is invariant under the vector R-rotation and $\overline{\Phi}\Phi$ is invariant under the phase rotation of Φ, the D-term is invariant under an arbitrary choice of vector R-charge. However, θ^2 has vector R-charge -2 (namely it transforms as $\theta^2 \mapsto \theta^2\,\mathrm{e}^{-2i\alpha}$). Thus, the F-term is invariant under vector R-rotation if and only if one can assign the vector R-charge of Φ so that $W(\Phi)$ has vector R-charge 2. This is the case when $W(\Phi)$ is a monomial. If

$$W(\Phi) = c\Phi^k, \tag{12.57}$$

then, by assigning vector R-charge $2/k$ to Φ, the F-term is made invariant under vector R-rotation. Namely, the system has a vector R-symmetry. In

such a case, the vector R-rotation of the superfield is realized as a transformation of the component field as

$$\phi \mapsto \mathrm{e}^{(2/k)i\alpha}\phi,\ \psi_\pm \mapsto \mathrm{e}^{((2/k)-1)i\alpha}\psi_\pm. \tag{12.58}$$

The conserved current is

$$J_V^0 = (2i/k)(\partial_0\overline{\phi}\phi - \overline{\phi}\partial_0\phi) - (2/k-1)(\overline{\psi}_+\psi_+ + \overline{\psi}_-\psi_-), \tag{12.59}$$

$$J_V^1 = (2i/k)(-\partial_1\overline{\phi}\phi + \overline{\phi}\partial_1\phi) + (2/k-1)(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-), \tag{12.60}$$

and the conserved charge is

$$F_V = \int J_V^0\, dx^0. \tag{12.61}$$

The vector R-rotation transforms the supercharges as

$$Q_\pm \mapsto \mathrm{e}^{-i\alpha}Q_\pm,\ \overline{Q}_\pm \mapsto \mathrm{e}^{i\alpha}\overline{Q}_\pm. \tag{12.62}$$

Also, the axial and vector R-rotations commute with each other.

12.2.2. Theory of a Twisted Chiral Superfield. One can also find a similar supersymmetric action for a twisted chiral superfield U. This time the action is expressed in the superspace as

$$S = -\int d^2x\, d\theta^4\, \overline{U}U + \left(\int d^2x d^2\widetilde{\theta}\, \widetilde{W}(U) \,+\, c.c.\right). \tag{12.63}$$

Note the minus sign in front of the D-term. This is required for the component fields to have the standard sign for the kinetic term. Chiral and twisted chiral superfields are related by the exchange of θ^- and $-\overline{\theta}^-$ which flips the sign for the D-term: $d\theta^- d\overline{\theta}^- = -d\overline{\theta}^- d\theta^-$ (the minus sign in $\theta^- \leftrightarrow -\overline{\theta}^-$ is for $\mathcal{Q}_- \leftrightarrow \overline{\mathcal{Q}}_-$). This last point enables us to borrow the formulae for a chiral superfield in finding the expression for the supersymmetry transformations, supercurrents, and R-symmetry generators in terms of the component fields. All we need to do is to make the replacements $\phi \to \upsilon$, $\psi_+ \to \overline{\chi}_+$, $\psi_- \to -\chi_-$, $F \to -E$, $\epsilon_+ \to -\overline{\epsilon}_+$, $Q_- \to \overline{Q}_-$ (or $G_-^\mu \to \overline{G}_-^\mu$), $F_V \to F_A$ and $F_A \to F_V$ with the others kept intact. For completeness we record here the relevant expressions. The supersymmetry transformation is

$$\begin{aligned}
\delta\upsilon &= \overline{\epsilon}_+\chi_- - \epsilon_-\overline{\chi}_+,\\
\delta\overline{\chi}_+ &= 2i\overline{\epsilon}_-\partial_+\upsilon + \overline{\epsilon}_+E,\\
\delta\chi_- &= -2i\epsilon_+\partial_-\upsilon + \epsilon_-E,\\
\delta E &= -2i\epsilon_+\partial_-\overline{\chi}_+ - 2i\overline{\epsilon}_-\partial_+\chi_-.
\end{aligned}$$